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Generalized Casimir operators of solvable Lie algebras with Abelian nilradicals*

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Abstract. A solvable complex Lie algebra L, of dimension N, with an Abelian nilradical of dimension r is shown to have precisely $2r - N$ generalized Casimir invariants (we always have $r \geqslant N/2$). They are constructed as invariants of the coadjoint representation of L and depend only on variables dual to elements of the nilradical. Their form, in general, involves logarithms of these variables in addition to rational and irrational functions. They give rise to genuine Casimir operators whenever they happen to be polynomials.

Résumé

Nous montrons qu'une algèbre de Lie L complexe résoluble de dimension N avec nilradical abélien de dimension r a précisément $m = 2r - N$ invariants de Casimir généralisés (on a toujours $r \geqslant N/2$). Ils sont calculés comme invariants de la représentation coadjointe de L et dépendent seulement de variables duales aux éléments du nilradical. Leur forme implique, en général, non seulement les fonctions rationnelles ou irrationnelles des variables, mais aussi des logarithmes. Ces invariants engendrent des vrais operateurs de Casimir seulement dans le cas où ce sont des polynômes.

1. Introduction

The purpose of this paper is to present some results on the Casimir invariants and generalized Casimir invariants of an n-dimensional solvable Lie algebra L over $\mathbb{C}$ with an Abelian nilradical (NR). Use will be made of a recent article [1] in which we obtained a classification of such Lie algebras and presented the general form of the commutation relations.

Casimir invariants (or Casimir operators) are polynomials in the enveloping algebra of a Lie algebra that commute with all elements of the Lie algebra. In other words, a Casimir invariant of a Lie algebra is an element of the centre of the enveloping algebra.

Casimir operators play a fundamental role in physics in that they represent important physical quantities in quantum mechanics such as angular momentum (the Casimir operator of O(3)), a relativistic elementary particle's mass and spin (Casimir operators of the Poincaré

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group) or the Hamiltonian of a particle undergoing geodesic motion (Casimir operator of the corresponding isometry group).

The Casimir operators of a Lie algebra L can be calculated directly as polynomials in the basis elements $X_i \in L$, commuting with all X_i. More efficiently, they can be calculated as invariants of the coadjoint representation of the corresponding Lie algebra [2, 3].

The Casimir operators of semi-simple Lie algebras are well known. Their number is equal to the rank of the considered Lie algebra [4–10]. Moreover, for a semi-simple Lie algebra L, all invariants of the coadjoint representation can be expressed as functions of $m(= \operatorname{rank} L)$ homogeneous polynomials.

For Lie algebras L that are not semi-simple, in particular for solvable Lie algebras, the situation is less clear. First of all, invariants of the coadjoint representation are not necessarily polynomials. They may be rational functions, or even irrational or transcendental ones. Their form and their number is, in general, not known.

Methods for calculating the polynomial and other invariants for arbitrary Lie algebras have been proposed [11–14]. One method is an infinitesimal one; it has been applied to low-dimensional Lie algebras [14], to subalgebras of the Poincaré Lie algebra [15] and to solvable Lie algebras with Heisenberg algebras as NRs [16]. Another method is a global one, making use of an explicit realization of the coadjoint representation of a Lie algebra [17]. This has been applied to affine Lie algebras (semi-direct sums of simple Lie algebras with Abelian ideals) [17].

In the representation theory of solvable Lie algebras, polynomial and non-polynomial invariants in the coadjoint representation appear on the same footing: they characterize irreducible representations. Casimir operators in the enveloping algebra correspond to polynomial invariants. The functions of the infinitesimal operators, corresponding to the non-polynomial invariants, will be called 'generalized Casimir operators'. In the study of the integrability of classical Hamiltonian systems, integrals of motion do not have to be polynomials in the dynamical variables [18, 19].

We feel that there is ample physical motivation for studying non-polynomial invariants on the same footing as polynomial ones.

2. Formulation of the problem and general results

2.1. Structure of the Lie algebra and its realization by differential operators

We are interested, in this paper, in finite-dimensional indecomposable solvable Lie algebras L with Abelian NRs [20, 21], considered over the field of complex numbers $\mathbb{C}$. In our previous paper [1] we have shown that such Lie algebras have the structure

$$L = F \dotplus \mathrm{NR} \qquad [F, F] \subseteq \mathrm{NR} \qquad [F, \mathrm{NR}] \subset \mathrm{NR} \qquad [\mathrm{NR}, \mathrm{NR}] = 0. \tag{2.1}$$

The subspace F is a factor-algebra, i.e. a Lie algebra modulo the nilradical. It is a Lie algebra only if we have $[F, F] = 0$.

We can always choose a basis

$$\{X_1, \ldots, X_f, N_1, \ldots, N_r\} \qquad f + r = N \qquad N = \dim L \tag{2.2}$$

such that the commutation relations are [1]

$$\begin{pmatrix} [X_\alpha, N_1] \\ \vdots \\ [X_\alpha, N_r] \end{pmatrix} = A^\alpha \begin{pmatrix} N_1 \\ \vdots \\ N_r \end{pmatrix} \qquad 1 \leqslant \alpha \leqslant f \leqslant r \tag{2.3a}$$

$$[N_i, N_k] = 0 \qquad i, k, j = 1, \ldots, r \tag{2.3b}$$

$$[X_\alpha, X_\beta] = R^j_{\alpha\beta} N_j \qquad \alpha, \beta = 1, \ldots, f$$

$$[A^\alpha, A^\beta] = 0 \tag{2.3c}$$

(thus $N_1, \ldots, N_r$ is a basis for the nilradical). For $f \geqslant 3$, the commuting matrices $A^\alpha \in \mathbb{C}^{r\times r}$, and the constants $R^j_{\alpha\beta}$, obey relations following from the Jacobi identities for the elements $\{X_\alpha, X_\beta, N_j\}$, namely

$$R^j_{\alpha\beta} A^\gamma_{jk} + R^j_{\gamma\alpha} A^\beta_{jk} + R^j_{\beta\gamma} A^\alpha_{jk} = 0. \tag{2.4}$$

The commuting matrices A_α are linearly nilindependent: no non-trivial linear combinations of these are nilpotent matrices. We shall call the matrices A^α the 'structure matrices'.

In order to calculate the generalized Casimir operators of the Lie algebra L, we shall work on the dual of L. We consider smooth functions

$$F(x_1, \ldots, x_f, n_1, \ldots, n_r) \tag{2.5}$$

where x_α and n_i are ordinary (commuting) variables on the space L^*, dual to L, and the differential operators $\widehat{N}_i$ and $\widehat{X}_\alpha$, realizing the coadjoint representation of L, are

$$\widehat{N}_i = -(A^\alpha)_{ik} n_k \partial_{x_\alpha} \tag{2.6}$$

$$\widehat{X}_\alpha = (A^\alpha)_{ik} n_k \partial_{n_i} + R^\alpha_{j\beta} n_j \partial_{x_\beta}. \tag{2.7}$$

It is easy to check that $\widehat{N}_i$ and $\widehat{X}_\alpha$ satisfy the same commutation relations as the Lie algebra elements n_i and x_α of equation (2.3).

The function F of equation (2.5) will be an invariant of the coadjoint representation of L if it satisfies the following linear first-order partial differential equations

$$\widehat{N}_i F = 0 \qquad i = 1, \ldots, r \tag{2.8a}$$

$$\widehat{X}_\alpha F = 0 \qquad \alpha = 1, \ldots, f. \tag{2.8b}$$

Our aim is to find a complete set of elementary solutions to equation (2.8). These elementary invariants will be called generalized Casimir invariants. Whenever they are polynomials, we can replace the variables x_α and n_i in F by the corresponding elements of the Lie algebra X_α and N_i and obtain, possibly after some symmetrization, an element of the centre of the enveloping algebra of L. Thus, generalized Casimir operators reduce to ordinary ones if they are polynomials.

2.2. *General form of the generalized Casimir invariants and their number*

Theorem 1. The solvable Lie algebra L over the field $\mathbb{C}$ satisfying the commutation relations (2.3) has exactly

$$m = r - f = 2r - N \tag{2.9}$$

functionally independent generalized Casimir invariants

$$C_i = C_i(n_1, \ldots, n_r) \qquad i = 1, \ldots, m \tag{2.10}$$

and they depend only on the variables n_i, dual to the elements of the nilradical $\mathrm{NR}(L)$.

Proof. The general invariant $F(x_1, \ldots, x_f, n_1, \ldots, n_r)$ must satisfy equation (2.8), in particular,

$$\widehat{N}_i F = -\sum_{\alpha=1}^{f}\sum_{k=1}^{r} A^{\alpha}_{ik} n_k \partial_{x_\alpha} F = 0 \qquad i = 1, \ldots, r. \tag{2.11}$$

Since the matrices A^α commute and are linearly nilindependent, they can be simultaneously transformed to their Kravchuk normal form [22, 23] (involving no nilpotent elements). As pointed out earlier [1], this means that the matrices A^α can all be simultaneously written in the block diagonal form

$$A^\alpha = \begin{pmatrix} T_1^\alpha(a_1^\alpha) & & & & & \\ & \ddots & & & & \\ & & T_p^\alpha(a_p^\alpha) & & & \\ & & & T_{p+1}^\alpha(0) & & \\ & & & & \ddots & \\ & & & & & T_{p+q}(0) \end{pmatrix} \tag{2.12a}$$

$$T_j^\alpha(a_j^\alpha) = \begin{pmatrix} a_j^\alpha & & 0 \\ & \ddots & \\ * & & a_j^\alpha \end{pmatrix} \in \mathbb{C}^{r_j \times r_j} \tag{2.12b}$$

$$a_j^\alpha = \begin{cases} \delta_j^\alpha & 1 \leqslant j \leqslant f \qquad \sum_{j=1}^{p+q} r_j = r,\ 1 \leqslant f \leqslant p \leqslant r \\ \in \mathbb{C} & f+1 \leqslant j \leqslant p \\ = 0 & p+1 \leqslant j \leqslant p+q \qquad 1 \leqslant \alpha \leqslant f,\ p+q \leqslant r \end{cases} \tag{2.12c}$$

(the star in equation (2.12*b*) denotes arbitrary entries). From equation (2.9) and (2.10) we see that we have

$$\begin{aligned} \widehat{N}_1 &= -n_1 \partial_{x_1} \\ \widehat{N}_{r_1+1} &= -n_{r_1+1} \partial_{x_2}, \ldots, \\ \widehat{N}_{r_1+r_2+\cdots+r_{f-1}+1} &= -n_{r_1+r_2+\cdots+r_{f-1}+1} \partial_{x_f}. \end{aligned} \tag{2.13}$$

Equations (2.11) and (2.13) imply that the invariant F, and hence the elementary invariants C_i, do not depend on $x_1, \ldots, x_f$, as indicated in equation (2.9).

Now, consider equation (2.8*b*) which simplifies to

$$\widehat{X}_\alpha F(n_1, \ldots, n_r) = \sum_{i,k=1}^{r} (A^\alpha)_{ik} n_k \partial_{n_i} F = 0 \qquad \alpha = 1, \ldots, f. \tag{2.14a}$$

Equation (2.14*a*) can be rewritten in matrix form as

$$M \begin{pmatrix} \partial_{n_1} F \\ \vdots \\ \partial_{n_r} F \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.14b}$$

From equation (2.12*a*) we see that the matrix M has maximal rank at every point of the space $N = \{n_1, \ldots, n_r\}$. Thus, we have f independent equations for a function of r variables. The number of independent solutions is, hence, $m = r - f$, as stated in equation (2.9).

This completes the proof. □

Notice that the structure constants $R^j_{\alpha\beta}$ of equation (2.3*b*) play no role in the calculation of the Casimir invariants.

We mention that theorem 1 is also valid over the field $F = \mathbb{R}$. The proof is quite analogous, but involves the usual complications due to the fact that the field $\mathbb{R}$ is not algebraically closed and, hence, the normal forms of commuting matrices can be more complicated.

The generalized Casimir invariants (2.10) are thus a complete set of $r - f$ functionally independent solutions of equation (2.14). They are best obtained by solving the corresponding characteristic system of ordinary differential equations

$$\frac{\mathrm{d}n_1}{A^\alpha_{1k}n_k} = \frac{\mathrm{d}n_2}{A^\alpha_{2k}n_k} = \ldots = \frac{\mathrm{d}n_r}{A^\alpha_{rk}n_k} \qquad \alpha = 1, \ldots, f. \tag{2.15}$$

The actual form of the invariants depends on the dimension f of the factor algebra F and on the specific form of the matrices A^α. The invariants are basis dependent.

Let us now consider some special cases in appropriate bases. They display all the characteristics of the general case.

3. Diagonal structure matrices

The simplest case, for any values of f and r, occurs when all the structure matrices A^α are diagonal (for an appropriate choice of the basis in the NR). By linear combinations of the elements $X_\alpha \subset F$, we can transform the structure matrices to the form

$$A_1 = \begin{pmatrix} 1 &&&&&& \\ & 0 &&&&& \\ && \ddots &&&& \\ &&& 0 &&& \\ &&&& a_{11} && \\ &&&&& \ddots & \\ &&&&&& a_{1r-f} \end{pmatrix}, \ldots, A_f = \begin{pmatrix} 0 &&&&&& \\ & \ddots &&&&& \\ && 0 &&&& \\ &&& 1 &&& \\ &&&& a_{f_1} && \\ &&&&& \ddots & \\ &&&&&& a_{fr-f} \end{pmatrix}. \tag{3.1}$$

The $r - f$ invariants can be chosen to be

$$I_k = n_{f+k} n_1^{-a_{1k}} n_2^{-a_{2k}} \ldots n_f^{-a_{fk}} \qquad k = 1, \ldots, r - f. \tag{3.2}$$

Thus, I_k are all rational if the eigenvalues a_i are rational.

4. Casimir invariants in the case $f = 1$

Let us consider the Lie algebra L over the field $\mathbb{C}$ for the case $f = 1$. The algebra L has a basis $\{X, N_1, \ldots, N_r\}$, where X is the only element of the basis not contained in the nilradical. The commutation relations in this case are

$$[X, N_i] = A_{ik}N_k \qquad [N_i, N_k] = 0. \tag{4.1}$$

By a change of basis in the NR and by multiplying X by a non-zero constant, we can always take the matrix A to its Jordan canonical form. We shall take each block as lower triangular and normalize its first eigenvalue to be $a_1 = 1$.

Thus we put

$$A = \begin{pmatrix} J_{r_1}(a_1) & & & & & \\ & \ddots & & & & \\ & & J_{r_m}(a_m) & & & \\ & & & J_{p_1}(0) & & \\ & & & & \ddots & \\ & & & & & J_{p_k}(0) \end{pmatrix} \tag{4.2a}$$

$$J_q(a) = \begin{pmatrix} a & & & & \\ 1 & a & & & \\ & & \ddots & & \\ & & \ddots & a & \\ & & & 1 & a \end{pmatrix} \in \mathbb{C}^{q\times q} \tag{4.2b}$$

$$a_1 = 1 \qquad a_j \neq 0 \qquad r_j \geqslant 1 \qquad p_i \geqslant 2 \qquad j = 1, \ldots, M \qquad i = 1, \ldots, K$$

$$M \geqslant 1 \qquad K \geqslant 0 \qquad \sum_{j=1}^{M} r_j + \sum_{i=1}^{K} p_i = r.$$

There will always be $(r-1)$ Casimir invariants and they satisfy just one partial differential equation, namely

$$\widehat{X}\, F(n_1, n_2, \ldots, n_r) = 0. \tag{4.3}$$

The operator $\widehat{X}$, representing $x \in L$, can be read from the matrix A. We have

$$\widehat{X} = \sum_{j=1}^{M} \left\{ a_j \sum_{k=1}^{r_j} n_{s_{j-1}+k} \partial_{n_{s_{j-1}+k}} + \sum_{k=2}^{r_j} n_{s_{j-1}+k-1} \partial_{n_{s_{j-1}+k}} \right\} + \sum_{k=1}^{K} \sum_{m=2}^{p_j} n_{t_{k-1}+m-1} \partial_{n_{t_{k-1}+m}} \tag{4.4}$$

where we have put

$$\begin{aligned} s_j &= \sum_{a=1}^{j} r_a \qquad s_0 = 0,\ j = 1, \ldots, M \\ t_k &= s_M + \sum_{b=1}^{k} p_b \qquad t_0 = s_M,\ k = 1, \ldots, K. \end{aligned} \tag{4.5}$$

Before treating the general case with A as in equation (4.2), let us first present two auxiliary results for cases when A is indecomposable.

Lemma 1. Consider the nilpotent Lie algebra $L_0 = \{X, N_1, \ldots, N_r\}$ with commutation relations as in equation (4.1) and with

$$A = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & & \ddots & 0 \\ & & \ddots & 1 & 0 \end{pmatrix} \in \mathbb{C}^{r\times r}. \tag{4.6}$$

The algebra L_0 has $r-1$ Casimir invariants that are all homogeneous polynomials, namely

$$I_1 = n_1 \tag{4.7a}$$

$$I_k = \sum_{j=0}^{k-2} \frac{k!}{j!}(-1)^j n_1^{k-1-j} n_2^j n_{k+1-j} + (-1)^{k-1}(k-1)n_2^k \qquad 2 \leqslant k \leqslant r-1. \tag{4.7b}$$

Proof. The operator $\widehat{X}$ of equation (4.4) simplifies to

$$\widehat{X}_0 = (n_1 \partial_{n_2} + \cdots + n_{r-1}\partial_{n_r}). \tag{4.8}$$

Clearly we have $\widehat{X}_0 I_k = 0$, $k = 1, \ldots, r-1$, for all expressions (4.7). To see that $I_1, \ldots, I_{r-1}$ are functionally independent, we calculate the Jacobian

$$J = \frac{\partial(I_1, \ldots, I_{r-1})}{\partial(n_1, \ldots, n_r)}. \tag{4.9}$$

It has the form

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 2n_3 & -2n_2 & 2!n_1 & 0 & 0 & \ldots & 0 \\ * & * & * & 3!n_1^2 & 0 & \ldots & 0 \\ * & * & * & * & 4!n_1^3 & \ldots & 0 \\ & & & & & \ddots & \vdots \\ * & * & & & * & \ldots & (r-1)!n_1^{r-2} \end{pmatrix} \tag{4.10}$$

so that we have rank $J = r-1$, as required. □

Lemma 2. Consider the solvable non-nilpotent Lie algebra $L = \{X, N_1, \ldots, N_r\}$ with commutation relations (4.1) and A as in (4.2*b*) with $q = r$. The $r-1$ generalized Casimir operators of L are

$$L_1 = a\frac{n_2}{n_1} - \ln(n_1) \tag{4.11a}$$

$$R_k = \frac{I_k}{(n_1)^k} \qquad k = 2, \ldots, r-1. \tag{4.11b}$$

Proof. The operator $\widehat{X}$ of equation (4.4) reduces to

$$\widehat{X} = D + \widehat{X}_0 \qquad D = \sum_{j=1}^{r} n_j \partial_{n_j} \tag{4.12}$$

with $\widehat{X}_0$ as in equation (4.8). We have

$$XL_1 = 0 \qquad XR_k = \frac{kI_k}{(n_1)^k} - \frac{kI_k}{(n_1)^k} = 0$$

so equations (4.11) defines invariants. The Jacobian J of equation (4.9) again has the triangular form (4.10) so that its rank is $r-1$. □

It is now clear that the invariants in the general case with A as in equation (4.2) will, in general, be of four types:

(i) invariants of the type (4.11), associated with indecomposable non-nilpotent Jordan blocks;

(ii) invariants of the type (4.7), associated with indecomposable nilpotent Jordan blocks;

(iii) invariants involving block 1 and the blocks $r_j, 2 \leqslant j \leqslant M$; and

(iv) invariants involving block 1 and block p_k, $1 \leqslant k \leqslant K$.

Indeed, to solve equation (4.3) we must solve the system of characteristic equations

$$\begin{aligned}\frac{\mathrm{d}n_1}{n_1} &= \frac{\mathrm{d}n_2}{n_1+n_2} = \ldots = \frac{\mathrm{d}n_{r_1}}{n_{r_1-1}+n_{r_1}} = \frac{\mathrm{d}n_{r_1+1}}{a_2 n_{r_1+1}} = \frac{\mathrm{d}n_{r_1+2}}{n_{r_1+1}+a_2 n_{r_1+2}} = \ldots \\ &= \frac{\mathrm{d}n_{r_1+r_2}}{n_{r_1+r_2-1}+a_2 n_{r_1}} = \ldots = \frac{\mathrm{d}n_{s_M+2}}{n_{s_M+1}} = \ldots = \frac{\mathrm{d}n_{s_M+p_1}}{n_{s_M+p_1-1}} = \ldots \\ &= \frac{\mathrm{d}n_{t_{k-1}+2}}{n_{t_{k-1}+1}} = \ldots = \frac{\mathrm{d}n_{t_{k-1}+p_k}}{n_{t_{k-1}+p_k-1}}. \qquad (4.13)\end{aligned}$$

We have r independent equations to solve. The strategy is:

(i) to solve equations within each block (this yields invariants of the type (4.7) or (4.11)); and

(ii) to connect all the blocks to block 1 in the simplest possible manner. We choose

$$\frac{\mathrm{d}n_{s_{j-1}+1}}{a_j n_{s_{j-1}+1}} = \frac{\mathrm{d}n_1}{n_1} \qquad \frac{\mathrm{d}n_{t_{k-1}+2}}{n_{t_{k-1}+1}} = \frac{\mathrm{d}n_1}{n_1} \qquad (4.14)$$

to obtain $M-1$ 'quasirational' invariants

$$Q_{s_{j-1}+1} = \frac{n_{s_{j-1}+1}}{(n_1)^{a_j}} \qquad j = 2, \ldots, M \qquad (4.15)$$

and K 'logarithmic' invariants

$$L_{t_{k-1}+2} = \frac{n_{t_{k-1}+2}}{n_{t_{k-1}+1}} - \ln n_1 \qquad k = 1, \ldots, K \qquad (4.16)$$

(s_j and t_k are defined in equation (4.5)). We call $Q_{s_{j-1}+1}$ quasirational because a_j is not necessarily a rational number. Even so, ratios of the type $Q_{s_{j-1}+1}/Q_{s_{m-1}+1}$ may be rational.

The results can now be summarized as a theorem.

Theorem 2. Let L be a solvable non-nilpotent Lie algebra satisfying the commutation relations (4.1) with A as in equation (4.2). The $r-1$ functionally independent invariants can be obtained as follows.

(i) Each non-nilpotent block $J_{r_j}(a_j)$ in A provides $(r_j - 1)$ invariants, namely

$$L_{s_{j-1}+2} = a_j \frac{n_{s_{j-1}+2}}{n_{s_{j-1}+1}} - \ln(n_{s_{j-1}+1}) \qquad (4.17a)$$

$$R_{s_{j-1}+\ell+1} = \frac{I_{s_{j-1}+\ell+1}}{(n_{s_{j-1}+1})^{\ell}} \qquad 1 \leqslant j \leqslant M,\ 2 \leqslant \ell \leqslant r_{j-1} \qquad (4.17b)$$

where $I_{s_{j-1+\ell+1}}$ is as in equation (4.7b), after an appropriate relabelling of n_i.

(ii) Each non-nilpotent block $J_{r_j}(a_j)$, combined with $J_{r_1}(1)$, provides one more invariant, namely $Q_{s_{j-1}+1}$ of equation (4.15).

(iii) Each nilpotent block $J_{p_k}(0)$ in A provides $p_k - 1$ polynomial invariants

$$P_{t_{k-1}+1} = n_{t_{k-1}+1} \tag{4.18a}$$

$$P_{t_{k-1}+\ell+1} = I_{t_{k-1}+\ell+1} \qquad 1 \leqslant k \leqslant K,\ 2 \leqslant \ell \leqslant p_k - 1. \tag{4.18b}$$

(iv) Each nilpotent block $J_{p_k}(0)$, combined with block $J_{r_1}(1)$, provides one further invariant, namely $L_{t_{k-1}+2}$ of equation (4.16).

Proof. The fact that expressions (4.15)–(4.18) provide invariants was proven above. That all $r - 1$ of them are functionally independent is seen by calculating the Jacobian (4.9). Its rank is $r - 1$. □

Corollary of theorem 1.

(i) If A is diagonal, then all invariants are of the type (4.15). They are polynomials if, and only if, all eigenvalues a_j are rational and of the same sign.

(ii) In all other cases, at least one of the invariants is logarithmic: one of the type (4.17*a*) for each $r_j \geqslant 2$ and one of the type (4.16) for each nilpotent block present.

Example. Consider the case with $r = 13$ when the matrix A has the form (4.2) with $M = 4$, $K = 2$, $r_1 = r_2 = 1$, $r_3 = 2$, $r_4 = 4$, $p_1 = 2$, $p_2 = 3$.

Invariants of the type (4.17*a*) come from blocks 2 and 3 while (4.17*b*) comes from block 4 only

$$L_4 = a_3 \frac{n_4}{n_3} - \ln n_3$$

$$L_6 = a_4 \frac{n_6}{n_5} - \ln n_5$$

$$R_7 = \frac{2n_5 n_7 - (n_6)^2}{(n_5)^2}$$

$$R_8 = \frac{6n_5^2 n_8 - 6n_5 n_6 n_7 + 2n_1^2}{(n_5)^3}.$$

Invariants of the type (4.15) are

$$Q_2 = \frac{n_2}{n_1^{a_2}} \qquad Q_3 = \frac{n_3}{n_1^{a_3}} \qquad Q_5 = \frac{n_5}{n_1^{a_4}}.$$

Invariants of the type (4.18) come from blocks 5 and 6 and are

$$P_9 = n_9 \qquad P_{13} = 2n_{11}n_{13} - n_{12}^2 \qquad P_{11} = n_{11}.$$

Invariants of the type (4.16) are

$$L_{10} = \frac{n_{10}}{n_9} - \ln n_1 \qquad L_{12} = \frac{n_{12}}{n_{11}} - \ln n_1.$$

Notice that the subscripts of the invariants go from 2 and 13 so that, for example, n_8 is first introduced in R_8.

5. Casimir invariants for low-dimensional nilradicals

In section 4, we considered the case of the largest possible nilradical of a solvable non-nilpotent Lie algebra, namely the case $f = 1$. Now we shall consider the opposite situation, when the factor algebra $F \sim L/\text{NR}$ is large, namely $f = r$, $r - 1$ and $r - 2$.

5.1. $f = r$

The case $f = r$ is only possible (over $\mathbb{C}$) if the algebra L satisfies $\dim L = 2$ or is decomposable into a direct sum of two-dimensional solvable Lie algebras [1]. The algebra L, in this case, has no Casimir invariants at all (see equation (2.9) of theorem 1).

5.2. $f = r - 1$

We have $r - 1$ linearly nilindependent commuting matrices $A_1, \ldots, A_{r-1} \in \mathbb{C}^{r\times r}$. Hence, they will form a subalgebra of a decomposable MASA of gl$(r, \mathbb{C})$. Only two decompositions of r are allowed by the condition of linear nilindependence, namely r one-dimensional blocks (i.e. all matrices A_i diagonalizable) or one two-dimensional block and $(r - 2)$ one-dimensional blocks.

In this case we have the following result.

Theorem 3. Let L be an indecomposable solvable Lie algebra of dimension $2r - 1$ with an Abelian nilradical of dimension r. Then L has precisely one generalized Casimir invariant that, in an appropriate basis, has one of the following two forms.

(i) If the matrices A_i of equation (2.3) are simultaneously diagonalizable, we have

$$I = n_1^{a_1} n_2^{a_2} \ldots n_{r-1}^{a_{r-1}} n_r^{-1}. \tag{5.1a}$$

(ii) If the matrices A_i are not simultaneously diagonalizable, we have

$$I = \frac{n_2}{n_1} - \ln(n_1^{a_1} n_3^{a_2} \ldots n_r^{a_{r-1}}). \tag{5.1b}$$

In case (i) the constants $a_\alpha \in \mathbb{C}$ are eigenvalues of the structure matrices A_α. In case (ii) they are the off-diagonal elements of the structure matrices.

Proof. It suffices to note that in case (i) all matrices A_α can be written as

$$A_\alpha = \text{diag}(b_{\alpha 1}, \ldots, b_{\alpha r-1}, a_\alpha) \qquad a_j \in \mathbb{C}, b_{\alpha k} = \delta_{\alpha k}. \tag{5.2}$$

Reading the operators $\widehat{X}_\alpha$ from equation (2.14), in this case we see that they all annihilate the invariant (5.1a).

In case (ii) we can choose

$$\begin{aligned} &A_\alpha = \text{diag}\left(\begin{pmatrix} b_\alpha & 0 \\ a_\alpha & b_\alpha \end{pmatrix}, c_{\alpha 3}, \ldots, c_{\alpha r}\right) \qquad \alpha = 1, \ldots, r - 1 \\ &b_\alpha = \delta_{\alpha 1} \qquad c_{\alpha j} = \delta_{j\alpha-1} \qquad a_\alpha \in \mathbb{C}. \end{aligned} \tag{5.3}$$

Again, all the corresponding operators $\widehat{X}_\alpha$ will annihilate the 'logarithmic' invariant I of equation (5.1b). □

5.3. $f = r - 2$

In this case, the matrices $A_\alpha \in \mathbb{C}^{r\times r}$ again form a decomposable MASA of $\mathrm{gl}(r, \mathbb{C})$ and four different decompositions are allowed. Let us state the results as a theorem.

Theorem 4. Let L be an indecomposable solvable Lie algebra of dimension $2r - 2$ with Abelian nilradical of dimension r. The structure matrices $\{A_\alpha\}$ and invariants $\{I_1, I_2\}$ have, in the appropriate basis, one of the following forms.
Case 1. Decomposition $r = r \times 1$

$$A_\alpha = \mathrm{diag}(c_{\alpha 1} \dots c_{\alpha,r-2}, a_\alpha, b_\alpha) \qquad c_{\alpha j} = \delta_{\alpha j} \qquad a_\alpha, b_\alpha \in \mathbb{C}.$$

Then

$$\begin{aligned} I_1 &= (n_1^{a_1} n_2^{a_2} \dots n_{r-2}^{a_{r-2}})(n_{r-1})^{-1} \\ I_2 &= (n_1^{b_1} n_2^{b_2} \dots n_{r-2}^{b_{r-2}})(n_r)^{-1}. \end{aligned} \tag{5.4}$$

Case 2. Decomposition $r = 2 + (r-2) \times 1$

$$A_\alpha = \mathrm{diag}\left[\begin{pmatrix} c_\alpha & 0 \\ y_\alpha & c_\alpha \end{pmatrix}, b_{\alpha 3}, \dots, b_{\alpha r-1}, a_\alpha\right]$$

$$c_1 = \begin{cases} 1 \\ 0 \end{cases} \qquad c_\alpha = 0 \qquad 2 \leqslant \alpha \leqslant r-2 \qquad a_\alpha \in \mathbb{C}$$

$$b_{\alpha j} = \delta_{j\alpha+1} \qquad j = 3, \dots, r-1 \text{ for } c_1 = 1$$

$$b_{\alpha j} = \delta_{j\alpha+1} \qquad j = 3, \dots, r \text{ for } c_1 = 0$$

where $y_\alpha \in \mathbb{C}$, at least one y_α satisfies $y_\alpha \neq 0$. For $c_1 = 1$ we have

$$\begin{aligned} I_1 &= (n_1^{a_1} n_3^{a_2} \dots n_{r-1}^{a_{r-2}})(n_r)^{-1} \\ I_2 &= \frac{n_2}{n_1} - \ln(n_1^{y_1} n_3^{y_2} \dots n_{r-1}^{y_{r-2}}). \end{aligned} \tag{5.5a}$$

For $c_1 = 0$ we have

$$\begin{aligned} I_1 &= n_1 \\ I_2 &= n_2 - n_1 \ln(n_3^{y_1} n_4^{y_2} \dots n_r^{y_{r-2}}). \end{aligned} \tag{5.5b}$$

Case 3. Decomposition $r = 2 + 2 + (r-4) \times 1$

$$A_\alpha = \mathrm{diag}\left[\begin{pmatrix} a_\alpha & 0 \\ y_\alpha & a_\alpha \end{pmatrix}, \begin{pmatrix} b_\alpha & 0 \\ z_\alpha & b_\alpha \end{pmatrix}, c_{\alpha 5}, \dots, c_{\alpha r}\right]$$

$$a_1 = \delta_{\alpha 1} \qquad b_\alpha = \delta_{\alpha 2} \qquad c_{\alpha_k} = \delta_{k,\alpha+2} \qquad 5 \leqslant k \leqslant r$$

where $y_\alpha, z_\alpha \in \mathbb{C}$. At least one y_α and one z_β satisfies $y_\alpha \neq 0$, $z_\beta \neq 0$.

In this basis the invariants are

$$
\begin{aligned}
I_1 &= \frac{n_2}{n_1} - \ln(n_1^{y_1} n_3^{y_2} n_5^{y_3} \dots n_r^{y_{r-2}}) \\
I_2 &= \frac{n_4}{n_3} - \ln(n_1^{z_1} n_3^{z_2} n_5^{z_3} \dots n_r^{z_{r-2}}).
\end{aligned}
\tag{5.6}
$$

Case 4. Decomposition $r = 3 + (r-3) \times 1$

$$
A_\alpha = \operatorname{diag}\left[\begin{pmatrix} a_\alpha & & \\ u_\alpha & a_\alpha & \\ y_\alpha & z_\alpha & a_\alpha \end{pmatrix}, b_{\alpha 4}, \dots, b_{\alpha r}\right]
$$
$$
a_\alpha = \delta_{\alpha 1} \qquad b_{k\alpha} = \delta_{\alpha+1} \qquad 4 \leqslant k \leqslant r.
$$

The complex constants u_α, y_α and z_α satisfy one of the following conditions.

(i) $u_\alpha = z_\alpha$ for all α. $u_\alpha = z_\alpha \neq 0$ for at least one value of α.

(ii) $u_\alpha = 0$ for all α. $y_\beta \neq 0$, $z_\gamma \neq 0$ for at least one value of β and γ.

(iii) $z_\alpha = 0$ for all $\alpha . y_\beta \neq 0$, $u_\gamma \neq 0$ for at least one value of β and γ.

The invariants in this basis are

$$
\begin{aligned}
I_1 &= \frac{n_2}{n_1} - \ln n_1^{u_1} n_4^{u_2} \dots n_r^{u_{r-2}} \\
I_2 &= y_{r-2}\frac{n_2}{n_1} - u_{r-2}\frac{n_3}{n_1} + \frac{1}{2} z_{r-2}\left(\frac{n_2}{n_1}\right)^2 \\
&\qquad + \ln[n_1^{(u_{r-2}y_1 - u_1 y_{r-2})} n_4^{(u_{r-2}y_2 - u_2 y_{r-2})} \dots n_{r-1}^{(u_{r-2}y_{r-3} - u_{r-3}y_{r-2})}].
\end{aligned}
\tag{5.7}
$$

Proof. The proof is the same in all cases. The operators $\widehat{X}_\alpha$ of equation (2.7) are read from the matrices A_α, given in theorem 4. It is then easy to verify that we have $\widehat{X}_\alpha I_1 = \widehat{X}_\alpha I_2 = 0$ for all α in all cases. It is obvious from the form of J_1 and J_2 that they are functionally independent. Conditions (i), (ii) or (iii) in case 4 are necessary to ensure commutativity of the matrices A^α. Cases (i), (ii) and (iii) correspond to Kravchuk signatures (111), (201) and (102), respectively. □

6. Conclusions

The main results of this paper can be summed up as follows.

(i) A solvable Lie algebra L, over the field $\mathbb{C}$, of dimension N, with an Abelian nilradical of dimension $r (r \geqslant N/2)$, has precisely $m = 2r - N$ generalized Casimir invariants. All of them are functions of the variables n_i, dual to the elements of the nilradical (theorem 1).

(ii) The generalized Casimir invariants, in general, involve logarithms of polynomials in n_i, as well as rational and irrational functions of n_i (theorems 2, 3 and 4).

(iii) Explicit expressions for the generalized Casimir operators were given for special cases when the dimension of the nilradical is equal or close to its minimal (or maximal) possible value.

(iv) The generalized Casimir operators are rational functions only in the very special case when all the structure matrices A^α of equation (2.3) are diagonal and the eigenvalues satisfy certain rationality conditions. Logarithmic expressions occur as soon as any of the structure matrices contain non-trivial Jordan blocks. The general form of the invariants is

$$I_k = \frac{P(n_1, \ldots, n_r)}{Q(n_1, \ldots, n_r)} + \ln(n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_r^{\alpha_r}) \tag{6.1}$$

where P and Q are homogeneous polynomials and α_i are complex constants. This was proven in all considered cases and we conjecture that the same is valid for all values of f and r and all standard forms of the structure matrices A_α.

Let us mention that the results depend heavily on the fact that the nilradical is Abelian. Indeed, for solvable Lie algebras with Heisenberg nilradicals, the invariants depend not only on the elements n_i but also on all elements of the Lie algebra [16]. Moreover, they are all rational and in some cases polynomial [16]. Similarly, the results will be quite different, for instance, in the case of the solvable Lie algebra of all (upper) triangular matrices.

As an example, take the Lie algebra of matrices

$$M = \begin{pmatrix} a_1 & x_{12} & x_{13} & x_{14} \\ & a_2 & x_{23} & x_{24} \\ & & a_3 & x_{34} \\ & & & a_4 \end{pmatrix} \tag{6.2}$$

($N = 10, r = 6$). Applying the methods used in this paper we find two invariants (one polynomial, the other rational), namely

$$\begin{aligned} I_1 &= a_1 + a_2 + a_3 + a_4 \\ I_2 &= \frac{(a_1 + a_4)x_{14} + x_{12}x_{24} + x_{13}x_{34}}{x_{14}}. \end{aligned} \tag{6.3}$$

Both depend on the elements of the nilradical x_{ik} and of the space L/NR, i.e. a_i.

To illustrate the complications arising over the field of real numbers, consider a four-dimensional Lie algebra of the form (2.3) with

$$A = \begin{pmatrix} a & 1 & \\ -1 & a & \\ & & b \end{pmatrix} \qquad a, b \in \mathbb{R}. \tag{6.4}$$

Obviously, A is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$. In agreement with theorem 1 (valid also for $F = \mathbb{R}$), we have two invariants, both depending on (n_1, n_2, n_3) only. However, their form is

$$\begin{aligned} I_1 &= (n_1^2 + n_2^2)^b / n_3^{2a} \\ I_2 &= \ln(n_1^2 + n_2^2) + 2a \tan^{-1}(n_2/n_1). \end{aligned} \tag{6.5}$$

Thus, in addition to rational and irrational functions and logarithms, we obtain inverse *trigonometric functions*.

The results and methods of this paper can be used to calculate the generalized Casimir invariants of any solvable Lie algebra L with an Abelian nilradical once the dimensions N and r are fixed. Moreover, the algorithm could be computerized in an efficient manner, making use of a differential Gröbner basis [24, 25], to solve the simultaneous sets of linear differential equations that arise.

Work on the classification of solvable Lie algebras with other types of nilradicals and their invariants is in progress.

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